Heron's Formula

For any triangle with sides a, b and c we define s to be half the perimeter thus

$$s = \frac{a+b+c}{2}.$$

Heron's formula for the area A of the triangle states

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

It is worth noting before we start that the conventional notation for $(\sin\theta)^2$ is $\sin^2\theta$ and that

$$\sin^2\theta + \cos^2\theta = 1 \text{ for all values of } \theta.$$

Proof Of Heron's Formula

It is worth saying that this is one of the ugliest ways to prove Heron's Formula, but it is the one which is most accessible to modern students whose geometry is weak (like myself).

The area of any triangle is given by $A = \frac{1}{2}ab\sin C$ where C is the angle between sides a and b. So

$$\begin{aligned} A &= \frac{1}{2}ab\sin C \\ &= \frac{1}{2}ab\sqrt{1-\cos^2 C}. \qquad \dagger \end{aligned}$$

Now using the cosine rule we know $c^2 = a^2+b^2-2ab\cos C$, so $\cos C = \frac{a^2+b^2-c^2}{2ab}$ and therefore

$$\cos^2 C = \frac{(a^2+b^2-c^2)^2}{(2ab)^2}.$$

Resisting the urge to expand the above and substituting this into † we obtain

$$\begin{aligned} A &= \frac{1}{2}ab\sqrt{1-\frac{(a^2+b^2-c^2)^2}{(2ab)^2}} \\ &= \frac{1}{2}ab\sqrt{\frac{(2ab)^2}{(2ab)^2}-\frac{(a^2+b^2-c^2)^2}{(2ab)^2}} \\ &= \frac{1}{2}ab\sqrt{\frac{(2ab)^2-(a^2+b^2-c^2)^2}{(2ab)^2}} \end{aligned}$$

Spotting the difference of two squares in the numerator and factoring we find

$$\begin{aligned} A &= \frac{1}{2}ab\sqrt{\frac{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}{(2ab)^2}} \\ &= \sqrt{\frac{(ab)^2}{4}\frac{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}{(2ab)^2}} \text{ (bringing } \frac{ab}{2} \text{ into } \surd) \\ &= \sqrt{\frac{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)}{16}} \\ &= \sqrt{\frac{((a+b)^2-c^2)(c^2-(a-b)^2)}{16}} \text{ (pair of difference two squares)} \\ &= \sqrt{\frac{(a+b+c)(a+b-c)(c+a-b)(c+b-a)}{16}} \\ &= \sqrt{\frac{a+b+c}{2}\cdot\frac{a+b-c}{2}\cdot\frac{a+c-b}{2}\cdot\frac{b+c-a}{2}} \\ &= \sqrt{\frac{a+b+c}{2}\cdot\frac{a+b+c-2c}{2}\cdot\frac{a+b+c-2b}{2}\cdot\frac{a+b+c-2a}{2}} \\ &= \sqrt{s(s-c)(s-b)(s-a)} \\ &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$